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LETTER TO THE EDITOR

U(1) Symmetry in Yangian and massless Thirring model

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Abstract. Yangian Y(sl(2)) constructed by fermionic field operators is extended to the q-deformed case which can be realized in the massless Thirring model. This leads to local U(1) gauge-invariance of Y(sl(2)) and evolution of the double-time Green function in q-statistics.

1. Introduction

Recently, many works in studying Yangian and its applications have been made including realizations of Yangian associated with sl(2), called Y(sl(2)), long-ranged interaction models, Yangian symmetry in the Hubbard model and so on [1–5]. A more interesting realization of Y(sl(2)) is in constructing the Yangian generators through fermions obeying the fermion commutation relations:

$$\begin{aligned} &[\chi_i(x,t), \chi_j^+(y,t)]_+ = \delta_{ij}\delta(x-y) \\ &[\chi_i(x,t), \chi_j(y,t)]_+ = [\chi_i^+(x,t), \chi_j^+(y,t)]_+ = 0 \qquad i,j = 1,2. \end{aligned} \tag{1}$$

Because of equation (1) the spin operators are easily constructed:

$$s_\alpha(x,t) = X^+(x,t)\frac{\sigma_\alpha}{2}X(x,t) \tag{2}$$

where $X^+(x,t) = (\chi_1^+(x,t), \chi_2^+(x,t))$ and $\sigma_\alpha(\alpha = 1,2,3)$ are Pauli matrices, such that s_α satisfy

$$[s_\alpha(x,t), s_\beta(y,t)] = \mathrm{i}\epsilon_{\alpha\beta\gamma}s_\gamma(x,t)\delta(x-y) \qquad \alpha,\beta,\gamma = 1,2,3. \tag{3}$$

With given $s_\alpha(x,t)$ it is straightforward to generate the generators of Y(sl(2)), I_α and J_α shown by Drinfeld [6–8]. However, for given Lie algebra

$$[I_\alpha, I_\beta] = \mathrm{i}\epsilon_{\alpha\beta\gamma}I_\gamma \tag{4}$$

the construction equation (2) giving rise to Y(sl(2)) does not need the commutation relations given by equation (1).

In this paper we shall show that for constructing Y(sl(2)) in this manner equation (1) can be extended to its q-deformed form which can be viewed as a special case of Zamolodchikov–Faddeev algebra or q-statistics. On the basis of the q-deformed fermionic commutation relations (see below) there exists a q-dependent correlation that vanishes when it reduces to the usual fermions. A physical example to be served in connection with the above q-algebra is the massless Thirring model.

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This paper is organized as follows. First, we get a new realization of Yangian with q-modification of the generators of Yangian algebra. Then we show that there exists local U(1) gauge-invariance of Yangian. Finally we compute a special correlation for the considered q-statistics.

2.

For sl(2) the Yangian Y(sl(2)) is generated by six generators I_α and J_α, $(\alpha = 1, 2, 3)$ [6]:

$$[I_\lambda, I_\mu] = c_{\lambda\mu\nu} I_\nu \qquad [I_\lambda, J_\mu] = c_{\lambda\mu\nu} J_\nu \tag{5}$$

$$[J_\lambda, [J_\mu, I_\nu]] - [I_\lambda, [J_\mu, J_\nu]] = h^2 a_{\lambda\mu\nu\alpha\beta\gamma}\{I_\alpha, I_\beta, I_\gamma\} \tag{6}$$

$$[[J_\lambda, J_\mu], [I_\sigma, J_\tau]] + [[J_\sigma, J_\tau], [I_\lambda, J_\mu]] = h^2 (a_{\lambda\mu\nu\alpha\beta\gamma} c_{\sigma\tau\nu} + a_{\sigma\tau\nu\alpha\beta\gamma} c_{\lambda\mu\nu})\{I_\alpha, I_\beta, J_\gamma\} \tag{7}$$

where

$$a_{\lambda\mu\nu\alpha\beta\gamma} = \frac{1}{4!} c_{\lambda\alpha\sigma} c_{\mu\beta\tau} c_{\nu\gamma\rho} c_{\sigma\tau\rho}$$

$$\{x_1, x_2, x_3\} = \sum_{i\neq j\neq k} x_i x_j x_k .$$

Setting $c_{\lambda\mu\nu} = \mathrm{i}\epsilon_{\lambda\mu\nu}$ $(\lambda, \mu, \nu = 1, 2, 3)$, both sides of equation (6) are identically zero, and we can recast equations (5) and (7) to [9]:

$$\begin{aligned} &[I_\alpha, I_\alpha] = 0 \qquad (\alpha = \pm, 3) \\ &[I_3, I_\pm] = \pm I_\pm \qquad [I_+, I_-] = 2I_3 \end{aligned} \tag{8}$$

$$\begin{aligned} &[I_3, J_\pm] = [J_3, I_\pm] = \pm J_\pm \\ &[I_+, J_-] = [J_+, I_-] = 2J_3 \end{aligned} \tag{9}$$

$$[J_3, [J_+, J_-]] = \frac{h^2}{4}(I_+ J_- - J_+ I_-) I_3 \tag{10}$$

with $I_\pm = I_1 \pm \mathrm{i} I_2$, $J_\pm = J_1 \pm \mathrm{i} J_2$. Equations (8)–(10) together with Jacobi identities yield equations (5) and (7), i.e. the relations for Y(sl(2)) consist in equations (8)–(10).

It can be checked that the generators of Y(sl(2)) can be written in the form

$$I_\alpha = \int \mathrm{d}x\, s_\alpha(x) \tag{11}$$

$$J_\alpha = T_\alpha + U J_\alpha^0 \tag{12}$$

$$T_\alpha = \int \mathrm{d}x\, X^+(x) \frac{\sigma_\alpha}{2} \partial_x X(x) \tag{13}$$

$$J_\alpha^0 = \frac{-\mathrm{i}}{2} \int\int \mathrm{d}x\, \mathrm{d}y\, \epsilon(y-x) \epsilon_{\alpha\beta\gamma} s_\beta(x) s_\gamma(y) \tag{14}$$

where U is a constant and s_α is defined by equation (2). Equations (11) and (12) obey Y(sl(2)), which can be referred to [10] or the continuous limit of Y(sl(2)) given by Uglov and Korepin [4]. Now, we start from the Zamolodchikov–Faddeev algebra [11]:

$$\begin{aligned} &\psi_i(x)\psi_j(y) =_{ij} R_{kl}(x-y)\psi_k(y)\psi_l(x) \\ &\psi_i^+(x)\psi_j^+(y) =_{ji} R_{lk}^*(y-x)\psi_k^+(y)\psi_l^+(x) \\ &\psi_i(x)\psi_j^+(y) =_{ki} R_{lj}(y-x)\psi_k^+(y)\psi_l(x) + \delta_{ij}\delta(x-y) \qquad i, j = 1, 2. \end{aligned} \tag{15}$$

Restricting the form of R-matrix as

$$R(x-y)=\begin{pmatrix}-1 & 0 & 0 & 0\\ 0 & {}_{12}R_{12}(x-y) & {}_{12}R_{21}(x-y) & 0\\ 0 & {}_{21}R_{12}(x-y) & {}_{21}R_{21}(x-y) & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \tag{16}$$

it is easy to check that in order to satisfy equation (3) on account of equation (15) the R-matrix should satisfy

$$\begin{aligned}&{}_{12}R_{12}(x-y)={}_{21}R_{21}(x-y)=0\\ &{}_{12}R_{21}(x-y)=-q=-\mathrm{e}^{\mathrm{i}\alpha} \qquad {}_{21}R_{12}(x-y)=-q^{-1}=-\mathrm{e}^{-\mathrm{i}\alpha}.\end{aligned}$$

Namely, when

$$\begin{aligned}&[\psi_j(x),\psi_j^+(y)]_+=\delta(x-y)\\ &[\psi_j(x),\psi_j(y)]_+=[\psi_j^+(x),\psi_j^+(y)]_+=0\\ &\psi_1^+(x)\psi_2^+(y)+q\psi_2^+(y)\psi_1^+(x)=0\\ &\psi_1^+(x)\psi_2(y)+q^{-1}\psi_2(y)\psi_1^+(x)=0 \qquad q=\mathrm{e}^{\mathrm{i}\alpha}\end{aligned} \tag{17}$$

the spin commutation relation equation (2) still works. It is easy to verify the identities in terms of equation (17):

$$\begin{aligned}&[\psi_i^+(x)\psi_i(x),\psi_j^+(y)]=\delta_{ij}\delta(x-y)\psi_j^+(y)\\ &[\psi_i^+(x)\psi_i(x),\psi_j(y)]=-\delta_{ij}\delta(x-y)\psi_j(y) \qquad i,j=1,2.\end{aligned}$$

Now we can prove that equations (11) and (12) still satisfy Y(sl(2)), provided $\psi_i(x,t)$ and $\psi_i^+(x,t)$ satisfy the q-deformed commutation relations equation (17). According to [10], we set

$$\begin{aligned}&\psi_1(x,t)=\mathrm{e}^{-\mathrm{i}\alpha\phi_2(x,t)}\chi_1(x,t)\\ &\psi_2(x,t)=\mathrm{e}^{\mathrm{i}\alpha\phi_1(x,t)}\chi_2(x,t)\end{aligned} \tag{18}$$

where

$$\phi_i(x,t)=\int_{-\infty}^{x}\mathrm{d}x\,\chi_i^+(x,t)\chi_i(x,t)=\int_{-\infty}^{x}\mathrm{d}x\,n_i(x,t)$$

and define $S_\alpha(x,t)$ through equation (2) with replacing $\chi(x,t)$ by $\Psi(x,t)$:

$$S_\alpha(x,t)=\Psi^+(x,t)\frac{\sigma_\alpha}{2}\Psi(x,t) \tag{19}$$

where $\Psi^+(x,t)=(\psi_1^+(x,t),\psi_2^+(x,t))$. Then we can regain equation (3) and equations (11)–(14) with $X(x,t)$ substituted by $\Psi(x,t)$. After careful calculation we can find that equations (8)–(10) still hold with redefined I_α, J_α, which means that the new operators constructed by the q-deformed fermions $\Psi(x,t)$ still satisfy the Y(sl(2)) relations.

To understand the physical meaning of the q-deformed relations of Y(sl(2)), let us return to the massless Thirring model. As pointed out in [10] the Hamiltonian

$$\begin{aligned}H=\mathrm{i}v\int\mathrm{d}x\,\{&\chi_1^+(x,t)\partial_x\chi_1(x,t)-\chi_2^+(x,t)\partial_x\chi_2(x,t)\\ &+2g\chi_1^+(x,t)\chi_2^+(x,t)\chi_2(x,t)\chi_1(x,t)\}\end{aligned}$$

becomes of diagonal form:

$$H=\mathrm{i}v\int\mathrm{d}x\,\{\psi_1^+(x,t)\partial_x\psi_1(x,t)-\psi_2^+(x,t)\partial_x\psi_2(x,t)\}$$

in terms of equation (18). Under this transformation the spin operators equation (2) undergo the corresponding transformation:

$$\begin{aligned} s^{+}(x,t) &\longrightarrow S^{+}(x,t) = \mathrm{e}^{\mathrm{i}\alpha\phi(x,t)} s^{+}(x,t) \\ s^{-}(x,t) &\longrightarrow S^{-}(x,t) = \mathrm{e}^{-\mathrm{i}\alpha\phi(x,t)} s^{-}(x,t) \\ s^{3}(x,t) &\longrightarrow S^{3}(x,t) = s^{3}(x,t) \end{aligned}$$

where $s^{\pm} = s_1 \pm \mathrm{i}s_2$ and $\phi(x,t) = \phi_1(x,t) + \phi_2(x,t) = \int_{-\infty}^{x} \mathrm{d}y\,(n_1(y,t) + n_2(y,t))$. It is obvious that the spin operators are subject to a local U(1) gauge transformation as equation (18) is made. We see that both $s_\alpha(x,t)$ and $S_\alpha(x,t)$ satisfy Y(sl(2)) the only difference being a local U(1) invariance under equation (18). This indicates that there exists local U(1) symmetry in such a realization for Y(sl(2)).

3.

If we turn to momentum representation:

$$H = \sum_k E_k(-a_k^{+}(t)a_k(t) + b_k^{+}(t)b_k(t))$$

where $a_k(t)$ and $b_k(t)$ correspond to $\psi_1(x,t)$ and $\psi_2(x,t)$ separately and $E_k (= vk)$ is determined by a Bethe ansatz [10], then we get commutation relations:

$$a_k^{+}(t)b_{k'}^{+}(t) + q b_{k'}^{+}(t)a_k^{+}(t) = 0. \tag{20}$$

Defining Bogoliubov–Zubalev double-time Green function:

$$G_{12}(t-t') = -\mathrm{i}\theta(t-t')\langle a_k^{+}(t)b_{k'}^{+}(t') + q^{-1}b_{k'}^{+}(t')a_k^{+}(t)\rangle \tag{21}$$

and noting that the r.h.s. vanishes as $q = 1$, i.e. for fermions. In comparison with equation (20) the r.h.s. of equation (21) is nontrivial. Following the standard treatment, we have

$$\mathrm{i}\frac{\partial}{\partial t}G_{12}(t-t') = \delta(t-t')(q^{-1}-q)\langle b_{k'}^{+}(t')a_k^{+}(t')\rangle + E_k G_{12}(t-t'). \tag{22}$$

Moreover, we get the evolution for $t \neq t'$:

$$\langle b_{k'}^{+}(t')a_k^{+}(t)\rangle = \frac{(q^{-1}-q)\mathrm{e}^{-\mathrm{i}E_k(t-t')}}{\mathrm{e}^{\frac{E_k}{\theta}} + q^{-1}}\langle b_{k'}^{+}(t')a_k^{+}(t')\rangle \tag{23}$$

which only exists for $q \neq 1$. For $t = t'$, equation (22) is an identity.

4. Conclusion

We have studied a new construction of Yangian Y(sl(2)) which is related to the q-statistics. In comparison with the original result, we find that there exists a local U(1) gauge-invariance of Yangian and we give an easily understandable explanation in terms of the massless Thirring model. It can probably be extended to more general examples.

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